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VARIABLE HEAT FLOW THROUGH WALLS OF CAVITY CONSTRUCTION, NATURALLY EXPOSED*

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Abstract—Exact solutions are calculated to two problems of non-steady heat flow in a cavity wall structure. The first considers periodic heat flow applied as a regular sinusoidal variation in the ambient temperature at the outside surface. The second examines transient flow in which the applied temperature condition is a single step-function of time. In both cases the opposite or inside surface of the wall is assumed to face a constant temperature enclosure.

INTRODUCTION

THE BOUNDARY CONDITIONS qualifying the flow of heat through the naturally exposed surfaces of a building assume a variety of different forms: rarely is the heat flux steady. A situation of practical interest, especially in the air-conditioning of buildings, is that of a harmonic variation of the outside ambient temperature combined with an indoor ambient temperature thermostatically controlled at a fixed value. For rigorous analysis the complex boundary conditions created by natural exposure have, perforce, to be simplified and a solution of the practical problem sought by examining an idealized case having as close a similarity to the actual situation as the analysis will permit. In settled weather the diurnal variation in the outside air temperature approximates to a regular wave form and may be described as steady periodic. On other occasions the heat flow is more appropriately considered as transient. Analytical solutions for both forms of boundary domain applied to an infinite plane slab of isotropic material are reported extensively in the literature [1, 2]. Little attention has been given to obtaining the corresponding formal analytical solutions defining heat flow in a cavity structure, although less exact methods of calculation have been devised. For example, Mackey and Wright [3], following Shklover [4], apply the concept of an equivalent homogeneous wall to obtain simple formulae for calculating steady periodic heat flow through composite structures. The properties of the homogeneous solid equivalent to the composite assembly of different conducting media, airspaces included, are derived empirically from a correlative examination of results calculated by more exact methods. A similar concept ("brick wall thickness of equivalent thermal capacity") is described by Bruckmayer [5] based on a more fundamental treatment due to Esser and Krischer [6] for calculating transient flow in multi-leaf walls. The application of matrix methods commonly used in electric circuit theory also allows numerical values for steady periodic temperatures in composite slabs to be calculated without difficulty [1].

Two heat conduction problems are solved below, using the Laplace transform method, to give the temperature distribution in either leaf of a single-cavity construction; solutions for certain other arrangements follow by taking limiting values of the parameters. The first case deals with periodic heat flow applied as a sinusoidal variation in the ambient temperature at the outside surface. The second examines transient flow in which the applied temperature condition is a single step-function of time.

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The method of analysis follows that similarly applied by Pratt and Ball [7], and Choudhury and Warsi [8] to the calculation of heat transmission in buildings though for other boundary conditions.

It is assumed that the heat flow is one-dimensional, and that the thermal properties, conductivity k and diffusivity K, of each slab are independent of temperature θ, position x and time t.

The prescribed boundary conditions refer to air temperature but this may be replaced, without loss of generality, with equivalent temperature* for representing surface heat exchange more conveniently in situations where the mean radiant temperature viewed by the surface is considered to be significantly different from the true air temperature.

1. PERIODIC HEAT FLOW THROUGH A CAVITY WALL

Consider the infinite plane slab of thickness l_1, thermal properties k_1 (conductivity), K_1 (diffusivity) representing the outer leaf, and thickness $(l_2 - l_1)$ of another, properties k_2, K_2 representing the inner leaf. A medium with thermal conductance h and negligible thermal capacity defines a closed airspace of uniform width sandwiched between the two leaves. At the positions $x = 0$ and $x = l_2$ heat is transferred between the wall surface and air at the respective rates h_0, h_i times their temperature difference. Initially the temperature differences throughout are zero. Subsequently the ambient temperatures at $x = 0$ and $x = l_2$ are maintained at $\theta_0 \sin \omega t$ and θ_i respectively. The situation considered has the following mathematical formulation.

Differential equations

$$K_1 \frac{\partial^2 \theta_1}{\partial x^2}(x, t) = \frac{\partial \theta_1}{\partial t}(x, t), \; 0 < x < l_i, \; t > 0 \tag{1.1}$$

$$K_2 \frac{\partial^2 \theta_2}{\partial x^2}(x, t) = \frac{\partial \theta_2}{\partial t}(x, t), \; l_1 < x < l_2, \; t > 0 \tag{1.2}$$

Boundary conditions, $t > 0$

$$-k_1 \frac{\partial \theta_1}{\partial x}(x, t) = h_0 \{\theta_0 \, e^{i\omega t} - \theta_1(x, t)\}, \; x = 0 \tag{1.3}$$

$$-k_1 \frac{\partial \theta_1}{\partial x}(x, t) = h \{\theta_1(x, t) - \theta_2(x, t)\}, \; x = l_1 \tag{1.4}$$

$$-k_2 \frac{\partial \theta_2}{\partial x}(x, t) = h \{\theta_1(x, t) - \theta_2(x, t)\}, \; x = l_1 \tag{1.5}$$

$$-k_2 \frac{\partial \theta_2}{\partial x}(x, t) = h_i \{\theta_2(x, t) - \theta_i(t)\}, \quad x = l_2 \tag{1.6}$$

Initial conditions

$$\left.\begin{aligned} \theta_1(x, 0) &= 0, \; 0 \leqslant x \leqslant l_1 \\ \theta_2(x, 0) &= 0, \; l_1 \leqslant x \leqslant l_2 \\ \theta_0(0) &= 0, \; x = l_1 \\ \theta_i(0) &= 0, \; x = l_2 \end{aligned}\right\} \tag{1.7}$$

* Equivalent temperature is the ambient temperature (hypothetical) at a surface that would give the same rate of heat flow through the surface as exists with the actual air temperature and radiation environment.

Applied temperature functions

$$\begin{aligned} \theta_0 &= 0, \ x = 0, \ t \leqslant 0 \\ &= \theta_0 \sin \omega t, \quad x = 0, \ t > 0 \\ \theta_i &= 0, \ x = l_2, \ t \leqslant 0 \\ &= \theta_i, \ x = l_2, \ t > 0 \end{aligned}$$

Let $\bar{\theta}_r(x, p)$, $(r = 1, 2)$, denote the Laplace transform of $\bar{\theta}_r(x, p)$, defined as

$$\bar{\theta}_r(x, p) = \int_0^\infty e^{-pt} \theta_r(x, t) \, dt \quad R(p) > 0$$

Applying the Laplace transform procedure to equation (1.1) through (1.6) it is found that the function $\bar{\theta}_r(x, p)$ satisfies the following subsidiary equations and boundary conditions;

Transformed differential equations

$$\frac{d^2\bar{\theta}_1}{dx^2}(x, p) - \frac{p\bar{\theta}_1}{K_1}(x, p) = 0, \ 0 < x < l_1 \tag{1.8}$$

$$\frac{d^2\bar{\theta}_2}{dx^2}(x, p) - \frac{p\bar{\theta}_2}{K_2}(x, p) = 0, \ l_1 < x < l_2 \tag{1.9}$$

Transformed boundary conditions

$$\frac{d\bar{\theta}_1}{dx}(x, p) + \frac{h_0}{k_1}\left\{\frac{\theta_0}{(p - i\omega)} - \bar{\theta}_1(x, p)\right\} = 0, \quad x = 0 \tag{1.10}$$

$$\frac{d\bar{\theta}_1}{dx}(x. p) + \frac{h}{k_1}\{\bar{\theta}_1(x, p) - \bar{\theta}_2(x, p)\} = 0, \quad x = l_1 \tag{1.11}$$

$$\frac{d\bar{\theta}_2}{dx}(x, p) + \frac{h}{k_2}\{\bar{\theta}_1(x, p) - \bar{\theta}_2(x, p)\} = 0, \quad x = l_1 \tag{1.12}$$

$$\frac{d\bar{\theta}_2}{dx}(x, p) + \frac{h_i}{k_2}\left\{\bar{\theta}_2(x, p) - \frac{\theta_i}{p}\right\} = 0, \quad x = l_2 \tag{1.13}$$

Attention will be confined to obtaining a solution for the temperature at the position $x = l_2$ after a long time when the sum of transients involving the initial condition has died away.

The system of equation (1.8) through (1.13) yields the result

$$\begin{aligned} \bar{\theta}_2(l_2, p) = {} & \frac{h_i\sqrt{K_2}\theta_i}{k_2 p f_2(p)}\left\{\sqrt{p}\cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) + \frac{h\sqrt{K_2}}{k_2}\sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1)\right\} + \\ & + \frac{h_0 h\sqrt{(K_1K_2)}\,p\theta_0}{k_1k_2(p - i\omega)F(p)} + \frac{h^2h_i\sqrt{(K_1)}\,K_2\theta_i}{k_1k_2^2F(p)f_2(p)}\left\{\sqrt{p}\cosh\sqrt{\left(\frac{p}{K_1}\right)}l_1 + \right. \\ & \left. + \frac{h_0\sqrt{K_1}}{k_1}\sinh\sqrt{\left(\frac{p}{K_1}\right)}l_1\right\} \end{aligned} \tag{1.14}$$

where

$$f_2(p) = 2\left(p + \frac{hh_iK_2}{k_2^2}\right)\sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) + 2(h + h_i)\sqrt{p}\frac{\sqrt{K_2}}{k_2}\cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) \tag{1.15}$$

and

$$F(p) = (h_0h + hh_i + h_0h_i)\,\frac{\sqrt{(K_1K_2)}}{k_1k_2}\,p\cosh\sqrt{\left(\frac{p}{K_1}\right)}\,l_1\cosh\sqrt{\left(\frac{p}{K_2}\right)}\,(l_2-l_1) +$$
$$+\,p\left(p + \frac{h_0hK_1}{k_1^2} + \frac{hh_iK_2}{k_2^2}\right)\sinh\sqrt{\left(\frac{p}{K_1}\right)}\,l_1\sinh\sqrt{\left(\frac{p}{K_2}\right)}\,(l_2-l_1) +$$
$$+\left\{p\,(h+h_i) + \frac{h_0hh_iK_1}{k_1^2}\right\}\frac{\sqrt{K_2}}{k_2}\,\sqrt{p}\sinh\sqrt{\left(\frac{p}{K_1}\right)}\,l_1\cosh\sqrt{\left(\frac{p}{K_2}\right)}\,(l_2-l_1) +$$
$$+\left\{p\,(h_0+h) + \frac{h_0hh_iK_2}{k_2^2}\right\}\frac{\sqrt{K_1}}{k_1}\,\sqrt{p}\cosh\sqrt{\left(\frac{p}{K_1}\right)}\,l_1\sinh\sqrt{\left(\frac{p}{K_2}\right)}\,(l_2-l_1) \qquad (1.16)$$

The Laplace inversion of the expression for $\bar{\theta}_2\,(l_2, p)$ gives the required solution as the sum of the residues at the simple poles represented by $p = 0$ and $p = i\omega$. The form of the result is

$$\theta_2\,(l_2, t) = \theta_2\,(l_2, t)_\omega + \theta_i\left\{1 - \frac{1}{h_i}\left(\frac{1}{h_0} + \frac{1}{h} + \frac{1}{h_i} + \frac{l_1}{k_1} + \frac{l_2-l_1}{k_2}\right)^{-1}\right\} \qquad (1.17)$$

defining a steady-state temperature distribution through the wall upon which is superimposed a steady periodic temperature function defined by

$$\theta_2\,(l_1, t)_\omega = \frac{hh_0\sqrt{(K_1K_2)}\theta_0\,e^{i\omega t}}{k_1k_2\,\{F(p)/p\}_{p=i\omega}} \qquad (1.18)$$

In manipulating equation (1.18) it is found convenient to rewrite the expression (1.16) for $F(p)$ so that

$$\frac{F(p)}{p} = \frac{1}{4\sqrt{p}}\,\{(\sqrt{p}+a_0)(\sqrt{p}+a_2)(\sqrt{p}+\overline{a+a_1})\,e^{\sqrt{p}\alpha} +$$
$$+\,(\sqrt{p}-a_0)(\sqrt{p}-a_2)(\sqrt{p}-\overline{a+a_1})\,e^{-\sqrt{p}\alpha} - (\sqrt{p}+a_0)(\sqrt{p}-a_2)(\sqrt{p}+\overline{a-a_1})\,e^{\sqrt{p}\beta} -$$
$$-\,(\sqrt{p}-a_0)(\sqrt{p}+a_2)(\sqrt{p}-\overline{a-a_1})\,e^{-\sqrt{p}\beta}\} \qquad (1.19)$$

in which

$$a_0 = h_0\,\frac{\sqrt{K_1}}{k_1}, \qquad a = h\,\frac{\sqrt{K_1}}{k_1}$$

$$a_1 = h\,\frac{\sqrt{K_2}}{k_2}, \qquad a_2 = h_i\,\frac{\sqrt{K_2}}{k_2}$$

$$\alpha = \frac{l_1}{\sqrt{K_1}} + \frac{(l_2-l_1)}{\sqrt{K_2}}, \qquad \beta = \frac{l_1}{\sqrt{K_1}} - \frac{(l_2-l_1)}{\sqrt{K_2}}$$

In (1.19), dropping the negligibly small term involving $e^{-\sqrt{p}\alpha}$, and making the substitutions as indicated leads finally to a solution of the form

$$\theta_2\,(l_2, t)_\omega = \frac{4a_0\,a_1\theta_0\,\sqrt{\omega}\,\sin(\omega t - \psi)}{\sqrt{(G^2 + H^2)}} \qquad (1.20)$$

where

$$\tan\psi = (H - G)/(H + G) \qquad (1.21)$$

with

$$\left.\begin{aligned} G &= [A \cos \alpha \sqrt{(\omega/2)} - B \sin \alpha \sqrt{(\omega/2)}]\, e^{\alpha\sqrt{(\omega/2)}} - [C \cos \beta \sqrt{(\omega/2)} - \\ &\quad - D \sin \beta \sqrt{(\omega/2)}]\, e^{\beta\sqrt{(\omega/2)}} - [E \cos \beta \sqrt{(\omega/2)} + F \sin \beta \sqrt{(\omega/2)}]\, e^{-\beta\sqrt{(\omega/2)}} \\ H &= [A \sin \alpha \sqrt{(\omega/2)} + B \cos \alpha \sqrt{(\omega/2)}]\, e^{\alpha\sqrt{(\omega/2)}} - [C \sin \beta \sqrt{(\omega/2)} + \\ &\quad + D \cos \beta \sqrt{(\omega/2)}]\, e^{\beta\sqrt{(\omega/2)}} + [E \sin \beta \sqrt{(\omega/2)} - F \cos \beta \sqrt{(\omega/2)}]\, e^{-\beta\sqrt{(\omega/2)}} \end{aligned}\right\} \quad (1.22)$$

and,

$$\begin{aligned} A &= -\omega \sqrt{(\omega/2)} + \{(a + a_1)(a_0 + a_2) + a_0a_2\} \sqrt{(\omega/2)} + (a + a_1)\, a_0a_2 \\ B &= \omega \sqrt{(\omega/2)} + (a_0 + a + a_1 + a_2)\, \omega + \{(a + a_1)(a_0 + a_2) + a_0a_2\} \sqrt{(\omega/2)} \\ C &= -\omega\, (\omega/2) + \{(a - a_1)(a_0 - a_2) - a_0a_2\} \sqrt{(\omega/2)} - (a - a_1)\, a_0a_2 \\ D &= \omega \sqrt{(\omega/2)} + (a_0 + a - a_1 - a_2)\, \omega + \{(a - a_1)(a_0 - a_2) - a_0a_2\} \sqrt{(\omega/2)} \\ E &= -\omega \sqrt{(\omega/2)} + \{(a - a_1)(a_0 - a_2) - a_0a_2\} \sqrt{(\omega/2)} + (a_0 - a_1)\, a_0a_2 \\ F &= \omega \sqrt{(\omega/2)} - (a_0 + a - a_1 - a_2)\, \omega + \{(a - a_1)(a_0 - a_2) - a_0a_2\} \sqrt{(\omega/2)} \end{aligned}$$

Two-layer wall

Equation (1.20) defines the steady cyclic temperature of the inside surface of a cavity wall facing a constant indoor temperature and exposed to a regular sinusoidal variation in the ambient temperature at the outside surface. The corresponding solution for a solid wall consisting of two layers (l_1, k_1, K_1) and ($\overline{l_2 - l_1}, k_2, K_2$) follows immediately by putting $1/h = 0$ in (1.20), giving

$$\theta_2\,(l_2, t)_\omega = \frac{4a_0 \sqrt{\omega}\theta_0}{\sqrt{(G^2 + H^2)}} \sin(\omega t - \psi) \quad (1.23)$$

where ψ, G and H are defined by the expressions in (1.21), (1.22) and the terms $A, \ldots, F$ now read

$$\begin{aligned} A &= (a_0 + a_2)(1 + b) \sqrt{(\omega/2)} + a_0a_2\,(1 + b) \\ B &= (a_0 + a_2)(1 + b) \sqrt{(\omega/2)} + \omega\,(1 + b) \\ C &= (a_2 - a_0)(1 - b) \sqrt{(\omega/2)} + a_0a_2\,(1 + b) \\ D &= (a_2 - a_0)(1 - b) \sqrt{(\omega/2)} - \omega\,(1 - b) \\ E &= (a_2 - a_0)(1 - b) \sqrt{(\omega/2)} - a_0a_2\,(1 - b) \\ F &= (a_2 - a_0)(1 - b) \sqrt{(\omega/2)} + \omega\,(1 - b) \end{aligned}$$

where, $b = (k_2/k_1) \sqrt{(K_1/K_2)}$, and a_0, a_2 are as defined above.

Numerical results

Calculated results of time-lag and amplitude ratio are given in Table 1 for a few cases selected to indicate the order of comparison between a solid brick wall and cavity walls incorporating brick and other material. The time-lag is defined as the interval between a sinusoidal variation in the ambient temperature at the outside surface and the temperature variation at the inside surface. The amplitude ratio measures the extent to which the simple harmonic variation in the outside ambient temperature is attenuated in its passage through the wall to the inside surface. It is interesting to note that the two alternative arrangements of such dissimilar materials as brick and timber differ, for the thicknesses specified, by no more than 0·7 h in the time-lag due to a diurnal variation of sinusoidal form in the outside ambient temperature. The difference will be even smaller for less dissimilar materials. An attempt to reduce the amount of computational work involved has been

Table 1. *Calculated values of time-lag and amplitude ratio of solid and cavity walls*

Wall (outside to inside)	Complete solution		Ignoring β terms	
	Time-lag (h)	Amplitude ratio	Time-lag (h)	Amplitude ratio
$4\frac{1}{2}$ in brick	3·7	0·324		
$4\frac{1}{2}$ in brick—cavity—1 in timber	5·0	0·114	4·4	0·112
1 in timber–cavity–$4\frac{1}{2}$ in brick	5·7	0·093	4·8	0·081
$4\frac{1}{2}$ in brick–cavity–3 in aerated concrete	6·7	0·073	5·5	0·071

made by repeating the calculations omitting the two terms that involve the β quantity. In the few cases examined the amplitude ratio is little changed by this approximation but the time-lag is underestimated by a significant amount.

2. TRANSIENT HEAT FLOW THROUGH A CAVITY WALL

The wall structure considered is that described above, with one set of thermal properties (subscript 1) in the region $0 < x < l_1$ denoting the outer leaf, and a second set (subscript 2) in the region $l_1 < x < l_2$ denoting the inner leaf. A closed airspace separates the two leaves. At $x = 0$, $x = l_2$ heat is transferred between the environment and the outside and inside surfaces of the wall at the respective rates h_0, h_i times their temperature difference. For all values of time the ambient temperature at the inside surface, $x = l_2$, is held constant at θ_i. Initially the system is in steady state. Subsequently the ambient temperature at the outside surface, $x = 0$, undergoes a step-function drop through θ_0 which is maintained indefinitely.

Stated formally it is required to calculate the temperature $\theta_r(x, t)$, $(r = 1, 2)$ that satisfies the following formulation:

Differential equations

$$K_1 \frac{\partial^2 \theta_1}{\partial x^2}(x, t) = \frac{\partial \theta_1}{\partial t}(x, t), \quad 0 < x < l_1, \quad t > 0 \tag{2.1}$$

$$K_2 \frac{\partial^2 \theta_2}{\partial x^2}(x, t) = \frac{\partial \theta_2}{\partial t}(x, t), \quad l_1 < x < l_2, \quad t > 0 \tag{2.2}$$

Boundary conditions

$$k_1 \frac{\partial \theta_1}{\partial x}(x, t) = h_0 \theta_1(x, t), \qquad x = 0, \quad t > 0 \tag{2.3}$$

$$k_1 \frac{\partial \theta_1}{\partial x}(x, t) = h\,\{\theta_2(x, t) - \theta_1(x, t)\}, \quad x = l_1, \quad t > 0 \tag{2.4}$$

$$k_2 \frac{\partial \theta_2}{\partial x}(x, t) = h\,\{\theta_2(x, t) - \theta_1(x, t)\}, \quad x = l_1, \quad t > 0 \tag{2.5}$$

$$k_2 \frac{\partial \theta_2}{\partial x}(x, t) = h_i\,\{\theta_i - \theta_2(x, t)\}, \qquad x = l_2, \quad t > 0 \tag{2.6}$$

Initial conditions

$$\left.\begin{aligned}\theta_1(x,0) &= \theta_0 + \frac{(1/h_0 + x/k_1)}{(1/h_0 + l_1/k_1 + 1/h + \overline{l_2 - l_1}/k_2 + 1/h_i)}(\theta_i - \theta_0)\\ \theta_2(x,0) &= \theta_0 + \frac{(1/h_0 + l_1/k_1 + 1/h + \overline{x - l_1}/k_2)}{(1/h_0 + l_1/k_1 + 1/h + \overline{l_2 - l_1}/k_2 + 1/h_i)}(\theta_i - \theta_0)\end{aligned}\right\} \tag{2.7}$$

Applied temperature functions

$$\begin{aligned}\theta_0 &= \theta_0, \quad x = 0 \quad t \leqslant 0\\ &= 0, \quad x = 0, \quad t > 0\\ \theta_i &= \theta_i, \quad x = l_2, \quad \text{all } t\end{aligned}$$

The method of solution follows that described above. An application of the Laplace transform gives:

Transformed differential equations

$$K_1 \frac{d^2\bar{\theta}_1}{dx^2}(x,p) - p\bar{\theta}_1(x,p) + \theta_0 + \frac{(1/h_0 + x/k_1)}{(1/h_0 + l_1/k_1 + 1/h + \overline{l_2 - l_1}/k_2 + 1/h_i)}(\theta_i - \theta_0) = 0. \tag{2.8}$$

$$K_2 \frac{d^2\bar{\theta}_2}{dx^2}(x,p) - p\bar{\theta}_2(x,p) + \theta_0 + \frac{(1/h_0 + l_1/k_1 + 1/h + \overline{x - l_1}/k_2)}{(1/h_0 + l_1/k_1 + 1/h + \overline{l_2 - l_1}/k_2 + 1/h_i)}(\theta_i - \theta_0) = 0. \tag{2.9}$$

Transformed boundary equation

$$k_1 \frac{d\bar{\theta}_1}{dx}(x,p) = h_0\bar{\theta}_1(x,p), \qquad x = 0, \tag{2.10}$$

$$k_1 \frac{d\bar{\theta}_1}{dx}(x,p) = h\,\{\bar{\theta}_2(x,p) - \bar{\theta}_1(x,p)\}, \qquad x = l_1, \tag{2.11}$$

$$k_2 \frac{d\bar{\theta}_2}{dx}(x,p) = h\,\{\bar{\theta}_2(x,p) - \bar{\theta}_1(x,p)\}, \qquad x = l_1, \tag{2.12}$$

$$k_2 \frac{d\bar{\theta}_2}{dx}(x,p) = h_i\left\{\frac{\theta_i}{p} - \bar{\theta}_2(x,p)\right\}, \qquad x = l_2 \tag{2.13}$$

The solution of (2.8) and (2.9) subject to (2.10) through (2.13) is

$$\begin{aligned}\bar{\theta}_1(x,p) = &-\frac{a_0\theta_0}{pf(p)}\left\{\cosh\sqrt{\left(\frac{p}{K_1}\right)}(l_1 - x) + \frac{a}{\sqrt{p}}\sinh\sqrt{\left(\frac{p}{K_1}\right)}(l_1 - x)\right\} -\\ &-\frac{a_0 a a_1\theta_0}{pf(p)\,g(p)}\left\{\cosh\sqrt{\left(\frac{p}{K_1}\right)}x + \frac{a_0}{\sqrt{p}}\sinh\sqrt{\left(\frac{p}{K_1}\right)}x\right\}\left\{\cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) +\right.\\ &\left.+\frac{a_2}{\sqrt{p}}\sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1)\right\} + \frac{R_1(x)}{R_{i\cdot 0}}\frac{(\theta_i - \theta_0)}{p} + \frac{\theta_0}{p}, \quad 0 < x < l_1\end{aligned} \tag{2.14}$$

$$\bar{\theta}_2(x,p) = -\frac{a_0 a_1 \theta_0}{p g(p)}\left\{\cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - x) + \frac{a_2}{\sqrt{p}}\sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - x)\right\} + \\ + \frac{R_{1\cdot 2}(x)}{R_{t\cdot 0}}\frac{(\theta_i - \theta_0)}{p} + \frac{\theta_0}{p},\ l_1 < x < l_2 \tag{2.15}$$

where

$$f(p) = \sqrt{p}\sinh\sqrt{\left(\frac{p}{K_1}\right)}l_1 + (a_0 + a)\cosh\sqrt{\left(\frac{p}{K_1}\right)}l_1 + \frac{a_0 a}{\sqrt{p}}\sinh\sqrt{\left(\frac{p}{K_1}\right)}l_1$$

$$g(p) = (p + a_0 a + a_1 a_2)\sinh\sqrt{\left(\frac{p}{K_1}\right)}l_1 \sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) + \\ + \left\{\sqrt{p}(a_1 + a_2) + \frac{a_0 a a_2}{\sqrt{p}}\right\}\sinh\sqrt{\left(\frac{p}{K_1}\right)}l_1 \cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) + \\ + \left\{\sqrt{p}(a_0 + a) + \frac{a_0 a_1 a_2}{\sqrt{p}}\right\}\cosh\sqrt{\left(\frac{p}{K_1}\right)}l_1 \sinh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1) + \\ + (a_0 a_1 + a_0 a_2 + a a_2)\cosh\sqrt{\left(\frac{p}{K_1}\right)}l_1 \cosh\sqrt{\left(\frac{p}{K_2}\right)}(l_2 - l_1).$$

$$a_0 = h_0\frac{\sqrt{K_1}}{k_1},\quad a = h\frac{\sqrt{K_1}}{k_1},\quad a_1 = h\frac{\sqrt{K_2}}{k_2},\quad a_2 = h_t\frac{\sqrt{K_2}}{k_2}$$

$$R_1(x) = 1/h_0 + x/k_1, \qquad 0 < x < l_1$$

$$R_{1\cdot 2}(x) = 1/h_0 + l_1/k_1 + 1/h + (x - l_1)/k_2, \quad l_1 < x < l_1$$

$$R_{t\cdot 0} = 1/h_0 + l_1/k_1 + 1/h + (l_2 - l_1)/k_2 + 1/h_t$$

The solution for the temperature distribution through the cavity wall is obtained from (2.14) and (2.15) in the usual way using the inversion theorem. The integrands are single-valued functions with simple poles at $p = 0$, $p = -\alpha_n^2$, and at $p = -\beta_m^2$, where α^2, $n = 1,2,\ldots$, are the positive roots of

$$\left\{\alpha_n(a_1 + a_2) - \frac{a_0 a a_2}{\alpha_n}\right\}\tan\frac{\alpha_n l_1}{\sqrt{K_1}} + \left\{\alpha_n(a_0 + a) - \frac{a_0 a_1 a_2}{\alpha_n}\right\}\tan\frac{\alpha_n(l_2 - l_1)}{\sqrt{K_2}} + \\ + \{(a_0 a + a_1 a_2) - \alpha_n^2\}\tan\frac{\alpha_n l_1}{\sqrt{K_1}}\tan\frac{\alpha_n(l_2 - l_1)}{\sqrt{K_2}} = a_0 a_1 + a_0 a_2 + a a_2 \tag{2.16}$$

and β_m, $m = 1, 2, \ldots$, are the positive roots of

$$\tan\frac{\beta_m l_1}{\sqrt{K_1}} = \frac{(a_0 + a)\beta_m}{\beta_m^2 - a_0 a}. \tag{2.17}$$

For the outer leaf, $0 < x < l_1, t > 0$

$$\theta_1(x, t) = \frac{R_1(x)}{R_{i \cdot 0}} \theta_i + 2a_0\theta_0 \sum_{m=1}^{\infty} \times$$

$$\frac{\{\cos \beta_m (l_1 - x)/\sqrt{K_1} + (a/\beta_m) \sin \beta_m (l_1 - x)/\sqrt{K_1}\} \exp[-\beta_m^2 t]}{\beta_m^2 \cdot \psi(\beta_m)} + 2a_0aa_1\theta_0 \sum_{m=1}^{\infty} \times$$

$$\frac{\{\cos \beta_m x/\sqrt{K_1} + (a_0/\beta_m) \sin \beta_m x/\sqrt{K_1}\} \{\cos \beta_m (l_2 - l_1)/\sqrt{K_2} + (a_2/\sqrt{K_2}) \sin \beta_m (l_2 - l_1)/\sqrt{K_2}\} \exp[-\beta_m^2 t]}{\beta_m^2 \{g(p)\}_{p=-\beta_m^2} \cdot \psi(\beta_m)} +$$

$$2a_0aa_1\theta_0 \sum_{n=1}^{\infty} \times$$

$$\frac{\{\cos \alpha_n x/\sqrt{K_1} + (a_0/\alpha_n) \sin \alpha_n x/\sqrt{K_1}\} \{\cos \alpha_n (l_2 - l_1)/\sqrt{K_2} + (a_2/\sqrt{K_2}) \sin \alpha_n (l_2 - l_1)/\sqrt{K_2}\} \exp[-\alpha_n^2 t]}{\alpha_n^2 \{f(p)\}_{p=-\alpha_n^2} \cdot \phi(\alpha_n)} \tag{2.18}$$

For the inner leaf, $l_1 < x < l_2, t > 0$

$$\theta_2(x, t) = \frac{R_{1 \cdot 2}(x)}{R_{i \cdot 0}} \theta_i +$$

$$2a_0a_1\theta_0 \sum_{n=1}^{\infty} \frac{\{\cos \alpha_n (l_2 - x)/\sqrt{K_2} + (a_0/\alpha_n) \sin \alpha_n (l_2 - x)/\sqrt{K_2}\} \exp[-\alpha_n^2 t]}{\phi(\alpha_n)} \tag{2.19}$$

where,

$$\psi(\beta_m) = \frac{1}{\beta_m}\left(1 + \frac{a_0 + a}{\sqrt{K_1}} + \frac{a_0a}{\beta_m^2}\right) \sin \frac{\beta_m l_1}{\sqrt{K_1}} + \frac{l_1}{\sqrt{K_1}}\left(1 - \frac{a_0a}{\beta_m^2}\right) \cos \frac{\beta_m l_1}{\sqrt{K_1}}$$

$$\begin{aligned}\phi(\alpha_n) = {} & [\{\alpha_n^2 (a_1 + a_2) - a_0aa_2\} l_1/\sqrt{K_1} + \{\alpha_n^2 (a_0 + a) - a_0a_1a_2\} (l_2 - l_1)/\sqrt{K_2}] \times \\ & \cos \alpha_n l_1/\sqrt{K_1} \, . \cos \alpha_n (l_2 - l_1)/\sqrt{K_2} + \\ & [\alpha_n (a_0 + a) + a_0a_1a_2/\alpha_n + \{\alpha_n (a_0a + a_1a_2) - \alpha_n^3\} l_1/\sqrt{K_1} + \\ & \alpha_n (a_0a_1 + a_0a_2 + aa_2) (l_2 - l_1)/\sqrt{K_2}] \cos \alpha_n l_1/\sqrt{K_1} \, . \sin \alpha_n (l_2 - l_1)/\sqrt{K_2} + \\ & [\alpha_n (a_1 + a_2) + a_0aa_2/\alpha_n + \{\alpha_n (a_0a + a_1a_2) - \alpha_n^3\} (l_2 - l_1)/\sqrt{K_2} + \\ & \alpha_n (a_0a_1 + a_0a_2 + aa_2) l_1/\sqrt{K_1}] \sin \alpha_n l_1/\sqrt{K_1} \, . \cos \alpha_n (l_2 - l_1)/\sqrt{K_2} - \\ & [2\alpha_n^2 + \{\alpha_n^2 (a_0 + a) - a_0a_1a_2\} l_1/\sqrt{K_1} + \{\alpha_n^2 (a_1 + a_2) - a_0aa_2\} (l_2 - l_1)/\sqrt{K_2}] \times \\ & \sin \alpha_n l_1/\sqrt{K_1} \, . \sin \alpha_n (l_2 - l_1) \sqrt{K_2}\end{aligned} \tag{2.20}$$

For practical purposes the particular solution $\theta_2(l_2, t)$ representing the temperature of the inside surface of the wall facing the constant temperature room is the most interesting. From (2.19)

$$\theta_2(l_2, t) = \frac{R_{1 \cdot 2}(l_2)}{R_{i \cdot 0}} \theta_i + 2a_0a_1\theta_0 \sum_{n=1}^{\infty} \frac{\exp[-\alpha_n^2 t]}{\phi(\alpha_n)} \tag{2.21}$$

Numerical calculation of the exact solution (2.21) is lengthy due largely to the labour involved in calculating the roots, α_n, defined by (2.16). An approximate and more easily calculable solution for $\theta_2(l_2, t)$ may be obtained by writing $g(p)$ as a polynomial with distinct zeros at $p = -\alpha_1^2, -\alpha_2^2, \ldots$, and representing this function as an infinite product of linear factors. The inversion $L^{-1}\{\bar{\theta}_2(l_2, p)\}$ then takes the form

$$L^{-1}\{\bar{\theta}_2(l_2, p)\} = \frac{R_{1.2}(l_2)}{R_{i.0}}\theta_i - \sum_{i=1}^{\infty} \frac{a_0 a_1 \theta_0 (p + \alpha_i^2)\exp[-\alpha_i^2 t]}{pC \prod_{j=1}^{\infty} (1 + p/\alpha_j^2)} \qquad i \neq j \tag{2.22}$$

where

$$C = a_0 a_1 + a_0 a_2 + a a_2 + a_0 a a_2 l_1/\sqrt{K_1} + a_0 a_1 a_2 (l_2 - l_1)/\sqrt{K_2}$$

Expanding $g(p)$ as a polynomial in p to $0(p^3)$ and inverting as indicated the solution for the temperature of the inside surface may be written approximately as

$$\theta_2(l_2, t) \simeq \frac{R_{1.2}(l_2)}{R_{i.0}}\theta_i + \frac{a_0 a_1 \theta_0}{C}\left(\frac{\alpha_2^2 \exp[-\alpha_1^2 t]}{\alpha_2^2 - \alpha_1^2} - \frac{\alpha_1^2 \exp[-\alpha_2^2 t]}{\alpha_2^2 - \alpha_1^2}\right) \tag{2.23}$$

in which $p = -\alpha_1^2, -\alpha_2^2$ are the roots of $g(p)$ expressed simply as a quadratic in p. In general, $\alpha_2^2 \geqslant \alpha_1^2$, so that for large values of t equation (2.23) reduces to

$$\theta_2(l_2, t) \simeq \frac{R_{1.2}(l_2)}{R_{i.0}}\theta_i + \frac{a_0 a_1 \theta_0 \exp[-\alpha_1^2 t]}{C} \tag{2.24}$$

It can be shown that $a_0 a_1/C$ simplifies to $(1/h_i)/R_{i.0}$; substituting this equivalent ratio of resistances in (2.24) the approximate solution may now be re-written in the more convenient form

$$\frac{\theta_2(l_2, 0) - \theta_2(l_2, t)}{\theta_0} \simeq \frac{1/h_i}{R_{i.0}}(1 - \exp[-\alpha_1^2 t]), \quad t \text{ large} \tag{2.25}$$

Equation (2.25) expresses the drop in the temperature of the inside surface as a fraction of the step-function change in the ambient temperature uniformly applied at the outside surface. The quantity $1/\alpha_1^2$ will be recognized as the time taken for the fractional change in the surface temperature to reach $(1 - 1/e)$ or 63 per cent of the ultimate steady state value represented as $(1/h_i)\,\theta_0/R_{i.0}$; $1/\alpha_1^2$ may be defined therefore as the time-constant of the wall for the problem considered. Writing $g(p)$ as a linear function and isolating p it can be shown that

$$\alpha_1^2 = \left(\frac{1}{2}\left\{W_1\frac{l_1}{k_1} + W_2\frac{(l_2 - l_1)}{k_2}\right\} + \frac{1}{R_{i.0}}\left\{\frac{W_1 + W_2}{h_i h_0} + \frac{W_1}{h_0}\left(\frac{1}{h} + \frac{l_2 - l_1}{k_2}\right) + \frac{W_2}{h_i}\left(\frac{1}{h} + \frac{l_1}{k_1}\right) - \frac{1}{3}\left[W_1\left(\frac{l_1}{k_1}\right)^2 + W_2\left(\frac{l_2 - l_1}{k_2}\right)^2\right]\right\}\right)^{-1} \tag{2.26}$$

where W_1, W_2 denote the respective heat capacities per unit area of the outer and inner leaves. The result in (2.26) is an approximate expression enabling α_1^2 to be calculated more readily than by graphical solution of the transcendental equation (2.15). The accuracy of the approximate method may be assessed by comparing numerical results given by (2.24) and the exact solution (2.19). Figure 1 shows the result of such a calculation applied to the case of an 11 in cavity wall of brick ($k_1 = k_2$, $K_1 = K_2$, $l_1 = l_2 - l_1$, $W_1 = W_2$) facing a constant temperature enclosure at the inside surface and exposed at the outside surface to a sudden drop through θ_0 in the ambient temperature. After an initial time interval the large difference between the results of the two methods begins to decrease and at a value of t equal to the time constant for the cavity wall the approximate solution agrees with the exact solution to within about 5 per cent. Thereafter the agreement is close.

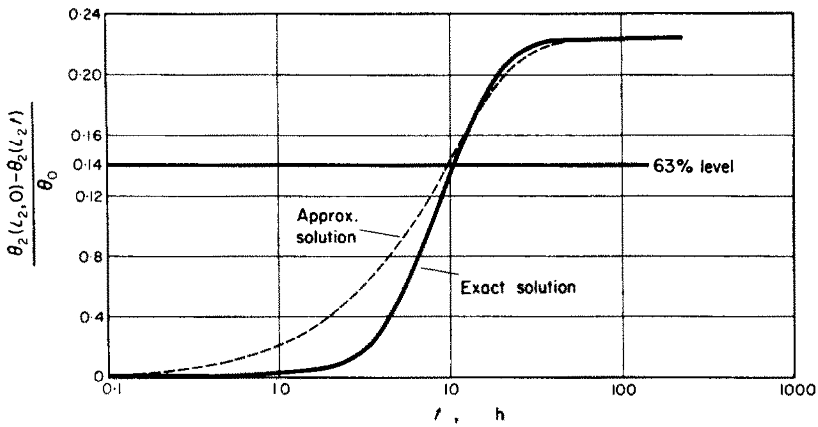

FIG. 1. Transient cooling of an 11 in cavity brick wall.

This result is worth consideration as providing possibly a simple method for comparing the transient response of different wall structures by calculating the time constant. It may be noted that if the boundary condition at the inside surface is changed from constant temperature to constant heat flux the time constant for homogeneous and composite structures is simply the ratio (heat stored/heat transmitted) in the steady state [7].

CONCLUSION

Analytical solutions have been obtained defining transient and periodic heat flow through walls of cavity construction composed of two layers of material of dissimilar thermal properties separated by a sealed airspace. The results may be used to calculate the flow of heat through walls naturally exposed at one surface and facing a constant ambient temperature at the other. Solutions for homogeneous and two-layer walls are included. The use of these results may perhaps be extended to include a three-layer solid wall in which a low thermal capacity slab of highly insulating material is sandwiched between dense material. This particular form of wall construction is an important feature of current developments in industrialized building.

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Résumé—On calcule les solutions exactes de deux problèmes de flux de chaleur transitoire dans une structure de paroi creuse. La première considère le flux de chaleur périodique appliqué comme une variation sinusoïdale régulière de la température ambiante à la surface extérieure. La seconde examine le flux transitoire dans lequel la condition de température appliquée est une fonction-échelon unique du temps. Dans les deux cas, on suppose que les surfaces exposée ou intérieure de la paroi sont en contact avec une enceinte à température constante.

Zusammenfassung—Für zwei Probleme des instationären Wärmestromes in einer Wand mit rauher Oberfläche werden exakte Lösungen berechnet. Die erste berücksichtigt einen periodischen Wärmestrom, der durch eine regelmässige sinusförmige Veränderung in der Umgebungstemperatur an der Aussenoberfläche aufgeprägt wird. Die zweite untersucht den momentanen Strom, für den die aufgeprägte Temperatur eine einzelne Stufenfunktion der Zeit ist. In beiden Fällen wird angenommen, dass die gegenüberliegende bzw. die Innenoberfläche der Wand ein Gebiet konstanter Temperatur darstellt.

Аннотация—Получены точные решения двух задач о нестационарном тепловом потоке в стенках полости. В первой рассматривается периодический тепловой поток, возникающий при синусоидальном изменении окружающей температуры на наружной поверхности. Во второй рассматривается неустановившийся поток, когда температура представляет собой один ступенчатый импульс во времени. В обоих случаях предполагается, что противоположная или внутренняя поверхность стенки находится при постоянной температуре.